HSE/Math in Moscow 2016-2017// Characteristic classes // Problems for discussion// Problem sheet 3

The cohomology of infinite Grassmannians

We now turn to the question of how to calculate the cohomology of Grassmann manifolds. Recall from problem sheet 1 on characteristic classes that for $m \leq n \leq \infty$ the space $G_m(\mathbb{R}^n)$ parametrises m-dimensional vector subspaces of $\mathbb{R}^n$. Similarly, $G_m(\mathbb{C}^n)$ parametrises m-dimensional vector subspaces of $\mathbb{C}^n$. We will also need the *orientable Grassmannian* $\tilde{G}_m(\mathbb{R}^n)$ which parametrises oriented m-dimensional vector subspaces of $\mathbb{R}^n$. These spaces come equipped with the tautological vector bundles which are denoted $\gamma^m(\mathbb{R}^n), \gamma^m(\mathbb{C}^n)$ and $\tilde{\gamma}^m(\mathbb{R}^n)$ respectively.

The bundle $\tilde{\gamma}^m(\mathbb{R}^n)$ is oriented (and $\gamma^m(\mathbb{R}^n)$ is not: it is not even orientable; why, by the way?). But these bundles are similar to $\gamma^m(\mathbb{R}^n)$ in that they are *universal oriented vector bundles*: Every real oriented vector bundle of rank m on a Hausdorff paracompact space (e.g. a CW-complex) X is isomorphic to $f^*(\tilde{\gamma}^m)$ for some $f : X \to \tilde{G}_m(\mathbb{R}^\infty)$. Moreover, the bundles $f^*(\tilde{\gamma}^m)$ and $g^*(\tilde{\gamma}^m)$ are isomorphic as oriented bundles iff $f, g : X \to \tilde{G}_m(\mathbb{R}^\infty)$ are homotopic.

There is an orientation-forgetting map $\tilde{G}_m(\mathbb{R}^n) \to G_m(\mathbb{R}^n)$ which is a covering map and, in fact, the universal cover. We will sometimes write G_m for $G_m(\mathbb{R}^\infty)$ or $G_m(\mathbb{C}^\infty)$ and $\tilde{G}_m$ for $\tilde{G}_m(\mathbb{R}^\infty)$. The space $\tilde{G}_m = \tilde{G}_m(\mathbb{R}^\infty)$ is also denoted $BSO(m)$. Later we will see why.

See e.g. Milnor-Stasheff, §§5, 12 and 14 for more details on how these spaces are constructed. In the sequel we need the following: $G_m(\mathbb{R}^n)$ has a CW-structure such that the number of cells of dimension r is the number of partitions of r into $\leq m$ integers each of which is $\leq n - m$. Note that this includes the case $n = \infty$ as well. See Milnor-Stasheff, ibid., Corollary 6.7. Similarly, $G_m(\mathbb{C}^n)$ has a CW-structure with no cells of odd dimension and such that the number of cells of dimension $2r$ is the number of partitions of r into $\leq m$ integers each of which is $\leq n - m$.

Question 1. Show that $H^*(G_m(\mathbb{R}^\infty), \mathbb{Z}/2)$ is the algebra of polynomials in $w_1(\gamma^m), \ldots, w_m(\gamma^m)$ over $\mathbb{Z}/2$. [Hint: induce $\gamma^1 \times \cdots \times \gamma^1$ on $\mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty$ from γ^m (see problem sheet 1 on characteristic classes) and show that the image of $H^*(G_m(\mathbb{R}^\infty), \mathbb{Z}/2)$ under the induced map

$$H^*(G_m(\mathbb{R}^\infty), \mathbb{Z}/2) \to H^*(\mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty, \mathbb{Z}/2) \tag{1}$$

contains the algebra of symmetric polynomials;deduce that all differentials in the cellular chain and cochain complexes of $G_m(\mathbb{R}^\infty)$ are zero, the map (1) is injective and its image is precisely the algebra of symmetric polynomials.]

Question 2. Show that $H^*(G_m(\mathbb{C}^\infty), \mathbb{Z})$ is the algebra of polynomials in $c_1(\gamma^m), \ldots, c_m(\gamma^m)$ over $\mathbb{Z}$. [Hint: one could use the strategy from the previous question, but with some modifications: this time all differentials are clearly zero, as all cells have even dimension, but the fact that the rank of the image of

$$H^n(G_m(\mathbb{C}^\infty), \mathbb{Z}) \to H^n(\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty, \mathbb{Z})$$

coincides with rank of the degree $n/2$ component of the algebra of symmetric polynomials over $\mathbb{Z}$ does not immediately imply that these groups coincide, as the former may contain the latter as a subgroup of finite index; on the other hand, the fundamental theorem of symmetric polynomials is true over any commutative ring.]

The Gysin sequence

Next we calculate the cohomology of $\tilde{G}_m = \tilde{G}_m(\mathbb{R}^\infty)$. We will only need to do this for rational coefficients. Even so, I don't know how to do this without spectral sequences. If you have a solution that does not use them, please let me know. The one given below uses the Gysin sequence, which is a particular case of the Leray spectral sequence in disguise.

Question 3. (a) Show that if $p : E \to X$ is an oriented vector bundle of rank r, then there is a long exact sequence

$$\cdots \longrightarrow H^i(X, \mathbb{Z}) \xrightarrow{\smile e(E)} H^{i+r}(X, \mathbb{Z}) \xrightarrow{p^*} H^{i+r}(_0, \mathbb{Z}) \longrightarrow H^{i+r}(X, Z) \xrightarrow{\smile e(E)} \cdots .$$

Here E_0 is E minus the zero section. This is the *Gysin exact sequence.* Clearly, in this sequence we can replace E_0 with the spherisation $S(E)$ of E with respect to some metric.

(b) Show that with coefficients mod 2 the above sequence exists even when E is not assumed orientable, if one replaces $e(E)$ by $e_{\mathbb{R}}(E)$ from problem sheet 2 on characteristic classes, question 4.

(c) If $\tilde{X} \to X$ is a double cover, construct a real line bundle $E \to X$ the spherisation of which is $\tilde{X}$. So applying part (b) we get a long exact sequence which involves the cohomology of $\tilde{X}$ and X.

Question 4. This question is for those who already know spectral sequences.

(a) Let $p : E \to X, r$ and E_0 be as in part (a) of the previous question. In particular, we again suppose that E is oriented. Show that the bundle $p : E_0 \to X$ is cohomologically simple and deduce the Gysin exact sequence from the Leray spectral sequence of $p : E_0 \to X$.

(b) Let us denote this spectral sequence $(E_s^{p,q})$. We now suppose X is path connected, so $E_2^{0,r-1} \cong \mathbb{Z}$. Let a be the positive generator of this group (specify what exactly this means). Show that $d_r(a) = e(E)$. [Hint: show that both these classes generate the same subgroup of $H^r(X,\mathbb{Z})$; what is this subgroup?]

Pontrjagin classes

If $E \to X$ is a real vector bundle of rank r then we define the *i-th Pontrjagin class $p_i(E)$ of E* as $(-1)^i c_{2i}(E \otimes \mathbb{C})$ and we set the *total Pontrjagin class of E* to be $1 + p_1(E) + \cdots + p_{[\frac{r}{2}]}(E)$. Note that these are invariants of real vector bundles which take values in integral cohomology.

Question 5. (a) Show that if $E \to X$ is a real vector bundle, then $E \otimes \mathbb{C}$ is a complex vector bundle which is isomorphic to its dual. Show that if E has rank 2, then $E \otimes \mathbb{C} = L \oplus L^*$ where L is a complex line bundle.

(b) Deduce from question 5 from problem sheet 2 on characteristic classes that all odd Chern classes of $E \otimes \mathbb{C}$ have order 2.

This at least partially explains why one discards odd Chern classes in the definition of the Pontrjagin classes. One could also ask what the sign $(-1)^i$ is doing there. The answer is: it makes some formulae nicer and some other less nice. We will see an example of the former in a minute but I suspect the main reason for including the sign is historical: it has always been there.

Question 6. Suppose $E, F \to X$ are two real vector bundles. Show that modulo elements of order 2 we have $p(E \oplus F) = p(E)p(F)$.

Question 7. (a) Show that if we take a complex vector bundle $E \to X$, then forget the complex structure and complexify it again, we get $E \oplus E^*$.

(b) Express the Chern classes of $E \oplus E^*$ is terms of the Chern classes of E.

(c) Show that the total Pontrjagin class of $\mathbb{C}P^n$ is $(1-a^2)^{n+1}$. Here $a \in H^2(\mathbb{C}P^n, \mathbb{Z})$ is the generator compatible with the complex structure.

Question 8. Can you find for every positive integer r a real vector bundle $E \to X$ of rank r such that none of the classes $c_1(E \otimes \mathbb{C}), \ldots, c_r(E \otimes \mathbb{C})$ is zero?

Question 9. Suppose $E \to X$ is an oriented real vector bundle of rank $r = 2s$.

(a) Show that $E \times \mathbb{C}$ is isomorphic to $E \oplus E$ or $E \oplus E$ with the orientation reversed, depending on whether $(-1)^{\frac{r(r-1)}{2}}$ is even or odd.

(b) Prove that $e(E)^2 = p_s(E)$.

Orientable infinite Grassmannians

Question 5. We want to show by induction on m that $H^*(\tilde{G}_m, \mathbb{Q})$ is a polynomial algebra in $p_1(\tilde{\gamma}^m), \ldots, p_{[\frac{m}{2}]}(\tilde{\gamma}^m)$ for odd m and $p_1(\tilde{\gamma}^m), \ldots, p_{[\frac{m-1}{2}]}(\tilde{\gamma}^m), e(E)$ for even m.

(a) Show that the space

$$\tilde{G}_1 = S^0 \subset S^1 \subset S^2 \subset \cdots$$

with the direct limit topology is contractible. This space is called *the infinite-dimensional sphere* and denoted S^∞.

(b) Let S be the spherisation of the vector bundle $\tilde{\gamma}_m$ on $\tilde{G}_m$ and let $p : S \to \tilde{G}_m$ be the projection. Show that S is also fibered over S^∞ and let q be the projection of that fibration. Show that if $x \in S^\infty$ then the fibre $S_x = q^{-1}(x)$ is homeomorphic to $\tilde{G}_{m-1}$. Deduce that S is homotopy equivalent to $\tilde{G}_{m-1}$. [Hint: one could use part (a) and the fact that the total space of a Serre fibration over a contractible space is homotopy equivalent to the fibre; why by the way?]

(c) Show that moreover, we can identify S_x with $\tilde{G}_{m-1}$ so that the bundle $p^*(\gamma_m)$ will split as $\varepsilon^1 \oplus \tilde{\gamma}_{m-1}$.

(d) Show that if m is even, then multiplication by the Euler class of γ^m is an injective map $H^*(\tilde{G}_m, \mathbb{Q}) \to H^{*+m}(\tilde{G}_m, \mathbb{Q})$. [Hint: to show this we only need to show the same result for at least one oriented bundle over at least one space; we already have seen such an example in this problem sheet.]

(e) A *commutative graded algebra* over a ring R with unit is a graded R-module $A^* = (A^n)_{n \in \mathbb{Z}}$ equipped with a multiplication which has the following properties: if $x \in A^n$ and $y \in A^m$, then $xy \in A^{n+m}$ and $xy = (-1)^{mn} yx$. In the sequel we will only consider graded algebras with a unit (which necessarily $\in A^0$) which are non-negatively graded, i.e. $A^{<0} = 0$. As an example one could take the cohomology ring $H^*(X, R)$ for an arbitrary space X. From now on R will be a field.

Suppose A^* and B^* are commutative non-negatively graded algebras over $\mathbb{Q}$. Suppose $m \in \mathbb{Z}_{>0}$ and an element $e \in A^m$ has the property from part (c): namely, multiplication by e is an injective map $A^* \to A^{*+m}$. Note that this implies that m is even, unless the characteristic of R is 2. Suppose that B^* is a polynomial algebra in $p_1, \ldots, p_k$ such that every p_i lives in some B^l. Suppose there is an exact sequence

$$0 \longrightarrow A^* \xrightarrow{\cdot e} A^{*+m} \longrightarrow B^* \longrightarrow 0$$

in which the map on the right is a homomorphism of commutative graded algebras. Show that A^* is a polynomial algebra in $\bar{p}_1, \cdots, \bar{p}_k, e$. Here $\bar{p}_i$ is a lift of $p_i \in B^l$ to A^l. [Hint: free algebras.]

(f) Deduce using part (e) and the Gysin sequence that if m is even, then $H^*(\tilde{G}_m, \mathbb{Q})$ is a polynomial algebra in $p_1(\tilde{\gamma}^m), \ldots, p_{\frac{m}{2}-1}(\tilde{\gamma}^m), e(\tilde{\gamma^m})$. Show also that $p_{\frac{m}{2}}(\tilde{\gamma}^m) = e(E)^2$.

(g) We suppose from now on that m is odd. Deduce from the Gysin sequence that $H^*(\tilde{G}_m, \mathbb{Q})$ is contained in $H^*(\tilde{G}_{m-1}, \mathbb{Q})$ as a ring and that

$$\dim H^i(\tilde{G}_{m-1}, \mathbb{Q}) = \dim H^i(\tilde{G}_m, \mathbb{Q}) + \dim H^{i-m}(\tilde{G}_m, \mathbb{Q}).$$

(h) We know by the induction hypothesis that the cohomology ring $H^*(G_{m-1}, Q)$ is a polynomial algebra in

$$p_1(\tilde{\gamma}^{m-1}), \ldots, p_{\frac{m-1}{2}-1}(\tilde{\gamma}^{m-1})$$

and $e(\tilde{\gamma}^{m-1})$; it also contains $p_{\frac{m-1}{2}}(\tilde{\gamma}^{m-1})$ but this class is $e(\tilde{\gamma}^{m-1})^2$. On the other hand, the ring $H^*(G_m, \mathbb{Q})$ contains $p_1(\tilde{\gamma}^m), \ldots, p_{\frac{m-1}{2}}(\tilde{\gamma}^m)$. Use this and part (g) to show that $H^*(G_m, \mathbb{Q})$ is a polynomial algebra in $p_1(\tilde{\gamma}^m), \ldots, p_{\frac{m-1}{2}}(\tilde{\gamma}^m)$.

Question 11. Using Question 3 (c) prove that $H^*(\tilde{G}_m, \mathbb{Z}/2)$ a polynomial algebra in $w_2(\tilde{\gamma}^m), \ldots, w_m(\tilde{\gamma}^m)$.
(b) Deduce that $e(\tilde{\gamma}^m)$ is non-zero if $m > 1$.
So in particular, the Euler class of an oriented vector bundle of odd rank need not be 0.

Note that it is also possible to calculate the cohomology of G_m and $\tilde{G}_m$ with integral coefficients. We will not need this, but if you're interested, you can have a look at the paper E.H. Brown, The cohomology of $BSO(n)$ and $BO(n)$ with integer coefficients, Proc. AMS 85 (1982), no. 2, 283-288.

The splitting principle for oriented real vector bundles

Question 12. In problem sheet 1 on characteristic classes we saw the splitting principle for complex vector bundles. There is a similar theorem for real vector bundles and cohomology mod 2. The proof is almost exactly the same as in the complex case. Further modifying the proof and using question 10 show the following version of the splitting principle:

Suppose $E \to X$ is a real oriented vector bundle of rank r. Then there is a space Y and a map $p : Y \to X$ such that the induced map $p^* : H^*(X, \mathbb{Q}) \to H^*(X, \mathbb{Q})$ is injective and

- $p^*(E) = E_1 \oplus \cdots \oplus E_{\frac{r}{2}}$ if r is even;
- $p^*(E) = E_1 \oplus \cdots \oplus E_{\frac{r-1}{2}} \oplus \varepsilon^1$ if r is odd.

Here E_i's are oriented rank 2 vector bundles on Y.

Note that it follows from question 5 (a) that for each i we have $E_i \otimes \mathbb{C} \cong L_i \oplus L_i^*$, where L_i are line bundles on Y.