An introduction to elliptic operators, HSE/MiM, 2016-2017 Problem sheet 4

Yet more linear algebra

Question 1. (a) For a real vector space V show that

$$\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}(V \otimes_{\mathbb{R}} \mathbb{C}, \mathbb{C}),$$

functorially in V.

(b) Let R be a commutative ring. [You may assume $R = \mathbb{R}$ if you like.] Show that for a free R-module F of finite rank and any R-module M we have

$$\mathrm{Hom}_R(F, M) \cong \mathrm{Hom}_R(F, R) \otimes_R M,$$

functorially in F. Does this remain true if the rank of F is not assumed finite?

So for a finite dimensional real vector space V we have

$$\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}(V \otimes_{\mathbb{R}} \mathbb{C}, \mathbb{C}),$$

functorially in V.

Since this is functorial, we have isomorphisms of vector bundles

$$T^*M \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{Hom}_{\mathbb{R}}(TM, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}(TM \otimes_{\mathbb{R}} \mathbb{C}, \mathbb{C}).$$

Here M is a smooth manifold. From now on we will be dropping subscripts in Hom's and tensor products when it is clear which ring is meant.

Note that part (b) above for general commutative rings is also useful: it can be used e.g. for proving Künneth type theorems for complexes modules of finite rank.

Almost complex manifolds

Let M be a smooth $2n$-manifold. Recall that an *almost complex structure* on M is a section J of the Hom bundle $\mathrm{Hom}(TM, TM)$ such that $J^2 = -I$. We then have a splitting $TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$ into eigenspaces of $J \otimes I$ which correspond to the eigenvalues i and $-i$. Similarly, we a decomposition for differential forms

$$T^*M \otimes \mathbb{C} \cong \mathrm{Hom}(TM \otimes \mathbb{C}, \mathbb{C}) = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M). \tag{1}$$

If $x \in M, v \in T_xM$, then for $\omega \in \Omega^{1,0}(M)$, respectively $\omega \in \Omega^{0,1}(M)$, we have $\omega(Jv) = i\omega(v)$, respectively $\omega(Jv) = -i\omega(v)$

Question 2. (a) Let $x \in M$ be a point. Show that an element $\omega \in T_x^*(M) \otimes \mathbb{C}$ annihilates every $v \in T_x^{0,1}(M)$ iff ω belongs to the fibre of $\Omega^{1,0}(M)$ over x. Similarly, show that an element $\omega \in T_x^*(M) \otimes \mathbb{C}$ annihilates every $v \in T_x^{1,0}(M)$ iff ω belongs to the fibre of $\Omega^{0,1}(M)$ over x.

(b) Show that complex conjugation on $TM \otimes \mathbb{C}$ interchanges the bundles $T^{1,0}M$ and $T^{0,1}M$ and similarly, complex conjugation on $T^*M \otimes \mathbb{C}$ interchanges $\Omega^{1,0}(M)$ and $\Omega^{0,1}(M)$.

Commutator of vector fields

Question 3. Let M be a smooth manifold.

(a) Recall that a vector field on M can be viewed both as a smooth section of TM and as a differential operator $C^\infty(M) \to C^\infty(M)$. Show that if V and W are two smooth vector fields on M, the operator

$$f \mapsto V(W(f)) - W(V(f)) \tag{2}$$

is again given by a vector field, which we will call the *commutator* of V and W and denote $[V, W]$. Note that one would normally expect (2) to be an operator of order 2, not 1.

(b) Now let ω be a differential 1-form on M and let V and W be two vector fields. Show that

$$V(\omega(W)) - W(\omega(V)) = d\omega(V, W) + \omega([V, W]).$$

Integrability conditions

Let M be a smooth $2n$-manifold with an almost complex structure J. The decomposition (1) induces a decomposition

$$\Lambda^k(T^*M) \otimes \mathbb{C} \cong \bigoplus_{p+q=k} \Omega^{p,q}(M)$$

with $\Omega^{p,q}(M) = \Lambda^p(\Omega^{1,0}(M)) \otimes \Lambda^q(\Omega^{0,1}(M))$. We set $\mathcal{E}^{p,q}(M) = \Gamma(\Omega^{p,q}(M))$. Elements of $\mathcal{E}^{p,q}(M)$ will be called *forms of type* (p,q) or (p,q)*-forms*. We extend the de Rham differential to $\Lambda^k(T^*M) \otimes \mathbb{C}$ by linearity and denote the result d.

Similarly, sections of $T^{1,0}(M)$ and $T^{0,1}(M)$ will be called *vector fields of type* $(1,0)$, respectively $(0,1)$. We extend the commutator operation to sections of $TM \otimes \mathbb{C}$ by linearity.

Recall that ∂ takes a form $\omega \in \mathcal{E}^{p,q}(M)$ to the $(p+1,q)$ component of $d\omega$. Similarly, $\bar{\partial}$ takes $\omega \in \mathcal{E}^{p,q}(M)$ to the $(p,q+1)$ component of $d\omega$.

Question 4. (a) Show that the commutator of two sections of $T^{1,0}(M)$ is again a section of $T^{1,0}(M)$ iff the commutator of two sections of $T^{0,1}(M)$ is again a section of $T^{0,1}(M)$.

(b) Show that for a form $\omega \in \mathcal{E}^{p,q}(M)$ we have $\partial\omega = \overline{\bar{\partial}\bar{\omega}}$. Deduce that $\bar{\partial}^2 = 0$ for all (p,q)-forms with $p+q=k$ iff $\partial^2 = 0$ for all (p,q)-forms with $p+q=k$.

(c) Show that each of the conditions of part (a) is equivalent to saying that the differential of a $(1,0)$-form only has $(2,0)$ and $(1,1)$-components, which in turn is equivalent to saying that the differential of a $(0,1)$-form only has $(0,2)$ and $(1,1)$-components. [Hint: use Question 3.]

Question 5. (a) Show that each of the conditions of Question 4 (c) is equivalent to saying that $d = \partial + \bar{\partial}$ on all forms. [Hint: every real valued form is locally a sum of products of 1-forms.]

(b) Show that if $d = \partial + \bar{\partial}$, then $\partial^2 = \bar{\partial}^2 = \bar{\partial}\partial + \partial\bar{\partial} = 0$.

(c) Show that if either ∂^2 or $\bar{\partial}^2 = 0$ on $\mathcal{E}^{0,0}(M)$ (we know that these conditions are equivalent by Question 4 (b)), then both conditions of 4 (c) are satisfied. [Hint: consider first the case $\omega = \partial f$ or $\bar{\partial} f$ for $f \in \mathcal{E}^{0,0}(M)$.]

So we deduce that the following properties are equivalent for an almost complex manifold (M, J):

- The commutator of two vector fields of type $(1,0)$ is again of type $(1,0)$.
- The commutator of two vector fields of type $(0,1)$ is again of type $(0,1)$.
- $d = \partial + \bar{\partial}$ on $(1,0)$-forms.
- $d = \partial + \bar{\partial}$ on $(0,1)$-forms.
- $d = \partial + \bar{\partial}$ on all forms.
- $\partial^2 = 0$ on $(0,0)$-forms.
- $\bar{\partial}^2 = 0$ on $(0,0)$-forms.
- $\partial^2 = 0$ on all forms.
- $\bar{\partial}^2 = 0$ on all forms.

And so on. There are many more conditions like these, and each of them is equivalent to saying that J is induced by a genuine complex analytic structure. Proving this in one direction is quite easy. Maybe we will do this in the next homework. The proof of the converse on the other hand requires some work. If one assumes that both M and J are real analytic, then there is a simple proof which can be found e.g. in §2.2 of Claire Voisin's book, vol. 1. If one does not assume this, then one has to prove it, which makes the job quite a bit more complicated.