An introduction to elliptic operators, HSE/MiM, 2016-2017 Problem sheet 4

The glueing lemma

We will shortly have to talk about Dirac operators. In fact, we have already seen a couple of examples, namely the de Rham and signature operators. To give more examples we will need structure groups and principle bundles. So we need a tool for constructing bundles of various kinds starting from local data. This tool is the *glueing lemma* and its versions can also be used to build topological spaces, sheaves, schemes and so on. This week's problems mainly (but not exclusively) fall into the category of boring but necessary.

Question 1. Suppose $\{X_i\}_{i\in I}$ is a collection of topological spaces and suppose $X_{ij} \subset X_i$ is an open subset. We suppose that $X_{ii} = X$ for all $i \in I$. For $i, j \in I$ let $\varphi_{ij} : X_{ij} \to X_{ji}$ be a homeomoprhism. We suppose that

- For all $i, j \in I$ we have $\varphi_{ji} = \varphi_{ij}^{-1}$ and $\varphi_{ii} = \mathrm{Id}_{X_i}$.
- For all $i, j, k \in I$ we have $\varphi_{ij}^{-1}(X_{jk}) = \varphi_{ik}^{-1}(X_{kj})$ and on this set $\varphi_{ki} \circ \varphi_{jk} \circ \varphi_{ij} = \mathrm{Id}$.

If these conditions are satisfied, then $((X_i), (X_{ij}), (\varphi_{ij}))$ is called *glueing data*.

(a) Set X to be the quotient of $\bigsqcup_{i\in I} X_i$ with respect to the equivalence relation generated by $x \sim \varphi_{ij}(x)$ for all $i, j \in I$ and all $x \in X_{ij}$. Show that each one of the natural maps $X_i \to X$ is a homeomorphism onto its image, and that the image is open in X.

(b) Now suppose that in addition to the above we are given a topological space F. We set $Y_i = X_i \times F$ and $Y_{ij} = X_{ij} \times F$. Suppose $A_{ij} : X_{ij} \to \mathrm{Homeo}(F)$ is a map such that the map $(x, f) \mapsto A_{ij}(x)(f)$ from Y_{ij} to F is continuous. We define then a map $\psi_{ij} : Y_{ij} \to Y_{ji}$ by

$$\psi_{ij}(x, f) = (\varphi_{ij}(x), A_{ij}(x)(f)).$$

What should we require from the maps A_{ij} if we want $((Y_i), (Y_{ij}), (\psi_{ij}))$ to be gluing data? Explain your answer and show that if this is the case, then the space Y obtained from $\bigsqcup_{i\in I} Y_i$ by identifying every $y \in Y_{ij}$ with $\psi_{ij}(y)$ will come equipped with a map $p : Y \to X$ such that (Y, X, p) is a locally trivial bundle.

Remark. Note that there is a small technical problem here. Instead of requiring $x \mapsto A_{ij}(x)$ to be a continuous map $X_{ij} \to \mathrm{Homeo}(F)$ we require $(x, f) \mapsto A_{ij}(x)(f)$ to be continuous. Here is why: suppose we have three topological spaces X, Y, Z. Set theoretically we have $\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z))$. One could ask if the same holds for continuous maps too, i.e., if the space $C(X \times Y, Z)$ is homeomorphic to $C(X, C(Y, Z))$ where $C(-, -)$ denotes the set of all continuous maps from X to Y with the compact-open topology. The answer is yes if X and Y are Hausdorff and Y is locally compact and no in general, see A. Hatcher, Algebraic Topology, Proposition A16.

A standard way of getting around this is to replace all spaces involved, including function spaces, with their compactly generated versions, see e.g. the Wikipedia article on compactly generated spaces. For locally compact spaces and CW-complexes this does not change anything, as they are already compactly generated; note that so are their open subsets as well. So when the base space X is a CW-complex or a topological manifold, we may as well require $A_{ij}(x)$ to be continuous as functions $X_{ij} \to \mathrm{Homeo}(F)$, with the target space equipped with its compactly generated topology.

Remark. Surprisingly, $\mathrm{Homeo}(F)$ with the compact-open topology is not always a topological group, but it is if F is Hausdorff and compact or Hausdorff and locally connected.

Remark. Suppose a topological group G acts on F on the left. Then passing to compactly generated spaces we get a continuous map $G \to \mathrm{Homeo}(F)$. So instead of constructing maps $X_{ij} \to \mathrm{Homeo}(F)$ one could construct maps $X_{ij} \to G$ which satisfy the conditions of part (b). If a bundle $E \to X$ is isomorphic to a bundle $E' \to X$ such that the functions A_{ij} can be chosen to take values in G rather then $\mathrm{Homeo}(F)$, then we say that the structure group of E admits a *reduction to* G. We'll talk more about this later.

Question 2. Suppose now that $E, F \to X$ are vector bundles. We choose an open cover $(X_i)_{i\in I}$ over elements of which both bundles trivialise, and we set $X_{ij} = X_i \cap X_j$ and let $\varphi_{ij} : X_{ij} \to X_{ji}$ be the identity map. Explain how the procedure from Question 1 can be used to rigorously construct the bundles $E \otimes F, \mathrm{Hom}(E, F) E^*$ and $\mathbb{P}(E)$.

Complex manifolds are integrable

Question 3. (a) Suppose now $(X_i)_{i\in I}, X_{ij}$ and φ_{ij} are as in Question 1 (a). We suppose that every X_i is a subset of $\mathbb{C}^n$ and every map φ_{ij} is holomorphic. Set $F = \mathbb{C}^k$ and suppose $A_{ij} : X_{ij} \to GL(\mathbb{C}^k)$ are holomorphic maps which satisfy the conditions of Question 1 (b). Construct a vector bundle $E \to X$ starting from this data and an almost complex structure on it. Recall that this is a section of $\mathrm{Hom}(E, E)$ which squares to minus identity.

(b) Now we let $k = n$ and let $A_{ij}(x)$ be the Jacobian matrix of $\varphi_{ij}(x)$ at x. Show that the resulting functions satisfy the conditions of Question 1 (a) and that the almost complex structure from part (a) is integrable. [Hint: do this first for open subsets of $\mathbb{C}^n$ and then globalise; you may use any of the integrability criteria from the previous problem sheet.]

Vector bundles on S^2

Question 5. Can you find a complex vector bundle on S^2 with non-zero first Chern class which would be trivial as a real vector bundle? Explain your answer.