HSE/Math in Moscow 2016-2017// Characteristic classes // Problems for discussion// Problem sheet 4

In this problem sheet we calculate the right hand side of the index formula for several classical differential operators. But first we need a little bit of K-theory.

The Grothendieck group

Question 1. Suppose that $(M, +)$ is a(n associative) commutative monoid. Recall that M has an identity element, which we denote 0. A typical example would be the set $\mathbb{Z}_{\geq 0}$ of non-negative integers with the addition operation. We set $K^0(M)$ to be the set of all couples $(m, n) \in M^2$ modulo the equivalence relation $(m_1, n_1) \sim (m_2, n_2)$ iff there is a $k \in M$ such that $m_1 + n_2 + k = m_2 + n_1 + k$.

(a) Show that this is indeed an equivalence relation. Show also that if $(m_1, n_1) \sim (m_2, n_2)$ and $(m'_1, n'_1) \sim (m'_2, n'_2)$, then $(m_1 + m'_1, n_1 + n'_1) \sim (m_2 + m'_2, n_2 + n'_2)$.

(b) Show that component-wise addition on M^2 induces an operation on $K^0(M)$ which is in fact a group operation. Show also that the map i which takes $m \in M$ to the class of $(m, 0)$ is a monoid homomorphism.

(c) Show that $K^0(M)$ is universal in the following sense: if G is an abelian group and $f : M \to G$ is a monoid homomorphism, then there is a unique group homomorphism $\bar{f} : K^0(M) \to G$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{i} & K^0(M) \\ & \underset{f}{\searrow} & \downarrow \bar{f} \\ & & G \end{array}$$

Show also that if $K^{0\prime}(M)$ is another abelian group with this property, then $K^0(M)$ is isomorphic to $K^{0\prime}(M)$, and the isomorphism is unique if one requires it to commute with the monoid maps from M.

Basically, this is just the construction of negative numbers in a fancy guise. If $M = \mathbb{Z}_{\geq 0}$, then (m, n) represents the number $m - n$.

Integers can not only be added, they can also be multiplied. Let us try to incorporate this in our construction. Suppose that M as above is equipped with another operation $\cdot$. As usual, we will often write mn for $m \cdot n$ for $m, n \in M$ and the operation $\cdot$ is often referred to as multiplication. The triple $(M, +, \cdot)$ is called a *semiring* iff $m(n + k) = mn + mk$ and $(n + k)m = nm + km$ and $0 \cdot m = m \cdot 0 = 0$ for all $m, n, k \in M$. We say that M is *commutative*, respectively *associative*, if $\cdot$ is commutative, respectively associative.

Question 2. Suppose $(M, +, \cdot)$ is a commutative associative semiring with a multiplicative identity 1.

(a) Mimicking the construction of multiplication of integers we introduce a multiplication on M^2:

$$(m_1, n_1) \cdot (m_2, n_2) = (m_1 m_2 + n_1 n_2, n_1 m_2 + m_1 n_2), m_1, m_2, n_1, n_2 \in M.$$

Show that this turns $K^0(M)$ into a commutative associative ring with the identity element being the equivalence class of $(1, 0)$. Show also that the map i from question 1 is a semiring homomorphism.

(b) State and prove the analogue of part (c) of the previous question with an abelian group G replaced by an associative commutative ring R with identity element.

We will not go into algebraic K-theory in any detail, but nevertheless we will mention a couple of examples.

Question 3. (a) Set M to be the set of isomorphism classes of finitely generated projective modules over a ring R. Then the group $K^0(M)$ is denoted $K^0(R)$. Show that if R is a field or $\mathbb{Z}$, then $K^0(R) \cong \mathbb{Z}$, with the isomorphism given by rank.

(b) Show however that if in part (a) one does not require the modules to be finitely generated, then one ends up with a monoid M such that $K^0(M) = 0$. Chances are, the argument you will find will be the same as the one discovered by S. Eilenberg in 1950's, hence the term *Eilenberg swindle*.

Now let X be a topological space and set M to be the set of isomorphism classes of complex vector bundles on X (of finite rank). The group $K^0(M)$ is then denoted $\mathcal{K}(X)$. If X is compact and Hausdorff, this group coincides with $K^0(X)$ which is defined as the set of homotopy classes of continuous maps $X \to BU$. We will prove this later. The groups $K^0(X)$ are more tractable, as they extend to a cohomology theory $K^*(-)$, while the groups $\mathcal{K}(X)$ are a bit mysterious for non-compact X. Note however that if X is paracompact and Hausdorff, e.g. a CW-complex, then there is a homomorphism $\mathcal{K}(X) \to K^0(X)$ which is functorial in X.

Note that if X is compact and Hausdorff, then R.S̃wan's theorem says that

$$\mathcal{K}(X) \cong K^0(X) \cong K^0(C(X, \mathbb{C}))$$

where $C(X, \mathbb{C})$ is the ring of all complex-valued continuous functions on X. If M is a smooth manifold, then one can replace continuous functions with smooth functions in this theorem.

Question 4. Prove that the Chern character map ch extends to a ring homomorphism $\mathrm{ch} : \mathcal{K}(X) \to H^{**}(X, \mathbb{Q})$.

We will also need relative K-groups. Given a topological pair (X, Y) we set M to be the monoid of all equivalence classes of triples (E, F, f) where E, F are finite rank complex vector bundles over X and $f : E|Y \to F|Y$ is an isomorphism; two couples (E, F, f) and (E', F', f') are equivalent iff there are isomorphisms $g : E \to E'$ and $h : F \to F'$ such that the following diagram commutes:

$$\begin{array}{ccc} E|Y & \xrightarrow{f} & F|Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ E'|Y & \xrightarrow{f'} & F'|Y \end{array}$$

The operation on M is component-wise direct sum. We set $\mathcal{K}(X, Y)$ to be the quotient of $K(M)$ by the subgroup generated by the classes of all triples (E, F, f) such that f extends to an isomorphism $E \to F$.

If (X, Y) is a CW-pair, then we can double X along Y. This means we can take two copies X_1, X_2 of X and glue them along Y. Let X' be the resulting space.

Question 5. (a) Supposing X, Y, X' are as above, prove that X' retracts onto Y.

(b) Prove that there is a map $\mathrm{ch} : \mathcal{K}(X, Y) \to H^{**}(X, Y)$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{K}(X, Y) & \longrightarrow & \mathcal{K}(X) \\ {\scriptstyle \mathrm{ch}}\downarrow & & \downarrow{\scriptstyle \mathrm{ch}} \\ H^{**}(X, Y, \mathbb{Q}) & \longrightarrow & H^{**}(X, \mathbb{Q}) \end{array}$$

where the top arrow takes a triple (E, F, f) as above to the class of E in $\mathcal{K}(X)$ minus the class of F.

The de Rham operator

Recall that $\tilde{\gamma}^m$ is the tautological oriented bundle over the infinite orientable Grassmannian $\tilde{G}_m = BSO(m)$. In this problem sheet we suppose that m is even and set $l = \frac{m}{2}$. Using question 12 from the previous problem sheet on characteristic classes we construct a CW-complex X and a map $q : X \to \tilde{G}_m$ which induces an injective map in rational cohomology and such that $q^*(\tilde{\gamma}^m)$ is a direct sum $E_1 \oplus \cdots \oplus E_l$ of oriented rank 2 bundles. By complexifying all bundles and using question 5 from ibid. we see that

$$q^*(\tilde{\gamma}^m \otimes \mathbb{C}) = L_1 \oplus L_1^* \cdots \oplus L_l \oplus L_l^*$$

where L_i are complex line bundles chosen so that $L_i \cong E_i$ as oriented real rank 2 bundles. We set $x_i = c_1(L_i)$ (so $c_1(L_i^*) = -x_i$).

Question 6. (a) Show that

$$\mathrm{ch}(\Lambda^{even}(q^*\tilde{\gamma}^m) \otimes \mathbb{C} - \Lambda^{odd}(q^*\tilde{\gamma}^m) \otimes \mathbb{C}) = \prod_{i=1}^{l} (1 - e^{x_i})(1 - e^{-x_i}).$$

(b) Show that the Euler class of $q^*(\tilde{\gamma}^m)$ is $\prod_{i=1}^{l} x_i$.

Finally, the total *Todd class* of a complex vector bundle $E \to X$ of rank r is defined as follows: express

$$\prod_{i=1}^{r} \frac{x_i}{1 - e^{-x_i}}$$

using elementary symmetric polynomials in $x_1, \ldots, x_r$ and substitute $c_i(E)$ for the i-th elementary symmetric polynomial. The resulting element of $H^{**}(X, \mathbb{Q})$ is the total Todd class $\mathrm{Td}(E)$ of E.

Question 7. Prove that

$$\mathrm{ch}(\Lambda^{even}(\tilde{\gamma}^m) \otimes \mathbb{C} - \Lambda^{odd}(\tilde{\gamma}^m) \otimes \mathbb{C}) \, \mathrm{Td}(\tilde{\gamma}^m \otimes \mathbb{C}) = (-1)^l (e(\tilde{\gamma}^m)^2.$$

Let us now equip $\tilde{\gamma}^m$ with a metric. We denote the resulting unit disk and unit sphere bundles D and S. Let p be the projection from the total space of $\tilde{\gamma}^m$ to $\tilde{G}_m$.

Question 8. Let ξ be an element of the fibre $\tilde{\gamma}_x$ of $\tilde{\gamma}^m$ over $x \in \tilde{G}_m$. Prove that $- \wedge \xi + i_\xi$ induces an isomorphism between $\Lambda^{even}(\tilde{\gamma}^m)$ and $\Lambda^{odd}(\tilde{\gamma}^m)$ pulled back to the fibre S_x of S over x. Here $- \wedge \xi$ is exterior multiplication by ξ and i_ξ is substitution of ξ into an element of $\Lambda^i(\tilde{\gamma}_x)$. Use this to construct an isomorphism σ between the pullbacks of $\Lambda^{even}(\tilde{\gamma}^m)$ and $\Lambda^{odd}(\tilde{\gamma}^m)$ to S.

Set $d = d(p^*\Lambda^{even}(\tilde\gamma^m) \otimes \mathbb{C}, p^*\Lambda^{odd}(\tilde\gamma^m) \otimes \mathbb{C}, \sigma)$. This is an element of $K(D,S)$ and we want to calculate $\mathrm{Th}^{-1}(\mathrm{ch}(d))$ where

$$\mathrm{Th} : H^*(\tilde G_m, \mathbb{Q}) \to H^*(D, S, \mathbb{Q})$$

is the rational Thom isomorphism given by $x \mapsto x \smile u$ with $u =$ the rational Thom class, see problem sheet 1 on characteristic classes.

Question 9. (a) Let $E \to X$ be an oriented real vector bundle of rank r with a metric. Let $D(E)$, respectively $S(E)$, be the unit disk bundle, respectively the unit sphere bundle. Show that the composite map

$$H^*(X) \overset{\mathrm{Th}}{\to} H^{*+r}(D(E), S(E)) \to H^*(D(E)) \overset{\cong}{\to} H^*(X)$$

is multiplication by the Euler class $e(E)$.

(b) Using part (a) and the fact that ch is a natural transformation show that

$$\mathrm{Th}^{-1}(\mathrm{ch}(d))e(\tilde\gamma^m) = \mathrm{ch}(\Lambda^{even}(\tilde\gamma^m) \otimes \mathbb{C} - \Lambda^{odd}(\tilde\gamma^m) \otimes \mathbb{C}).$$

Question 10. Prove using the fact that $H^*(\tilde G_m, \mathbb{Q})$ has no zero-divisors that

$$\mathrm{Th}^{-1}(\mathrm{ch}(d))\,\mathrm{Td}(\tilde\gamma^m \otimes \mathbb{C}) = (-1)^l e(\tilde\gamma^m).$$

Now recall that the Atiyah-Singer index theorem gives the following recipe for calculating the index of an elliptic differential operator $D : \Gamma(E)\Gamma(F)$, where E, F are smooth complex vector bundles over a smooth compact orientable manifold M of dimension m: let $p : T^*M \to M$ be the projection. We equip M with a Riemannian metric and let $D(T^*M)$ and $S(T^*M)$ be the corresponding unit disk bundle and unit sphere bundle respectively. The symbol $\sigma = \sigma_D$ of D restricted to $S^*(T^*M)$ gives us an isomorphism $\sigma : p^*(E) \to p^*(F)$. This isomorphism in turn gives us an element d_D in $K*(D(T^*M), S(T^*M))$, namely

$$d_D = d(p^*E, p^*F, \sigma).$$

We then have to evaluate the $\mathrm{Th}^{-1}(d_D)\,\mathrm{Td}(T^*M \otimes \mathbb{C})$ on the fundamental class of M:

$$\mathrm{index}\, D = (-1)^{\frac{m(m+1)}{2}} \int_M \mathrm{Th}^{-1}(d_D)\,\mathrm{Td}(T^*M \otimes \mathbb{C}).$$

Note that D participates in this formula only via its symbol σ, which is purely topological object: namely, it is a section of the Hom bundle $\mathrm{Hom}(p^*E, p^*F)$.

Question 11. We know that if $E = \Lambda^{even}(T^*M), F = \Lambda^{odd}(T^*M)$ and $D = d + d^*$, then the symbol σ of D at $\xi \in T^*_x M$ is given by

$$- \wedge \xi + i_\xi.$$

Prove that the index of D is equal $\int_M e(T^*M)$.

The signature operator

So we see that the index theorem implies that the Euler characteristic of an orientable even-dimensional manifold is the Euler class evaluated on the fundamental class of M. This is reassuring, but a bit boring: it is not too hard to prove this without the index theprem, see e.g. Milnor-Stasheff, Characteristic classes, Corollary 11.12.

Question 12. Let V be a finite-dimensional vector space over a field k of characteristic $\neq 2$ and let $q : V \to k$ be a quadratic form. Prove that the Clifford algebra $\mathrm{Cl}(V,q)$ is isomorphic as a vector space to the exterior algebra Λ^*V functorially in V (in particular, the isomorphism depends only on V and not on q).

Question 13. Let k, V, q be as above. You may assume that $k = \mathbb{R}$ and q is positive definite. Set $n = \dim_k V$. Using the isomorphism from question 1, describe the Hodge star operator

$$* : \Lambda^i(V) \to \Lambda^{n-i}V$$

in terms of the Clifford multiplication. [Hint: choose an orthogonal basis and consider the product of all its elements; there might be sign issues to take care of.]

Question 14. Let k be as above and let (V, q_1) and (W, q_2) be finite-dimensional k-vector spaces equipped with quadratic forms q_1, q_2. Let $q_1 \oplus q_2$ be the direct sum of q_1 and q_2. Set $n = \dim_k V, m = \dim_k W$ and let $*_V, *_W, *_{V\oplus W}$ be the Hodge star operators on V, W and $V \oplus W$ equipped with the quadratic forms q_1, q_2 and $q_1 \oplus q_2$ respectively. We assume that both m and n are even.

(a) Prove that

$$*_{V\oplus W}(x \wedge y) = (-1)^{ij} *_V(x) \wedge *_W(y)$$

for all $x \in \Lambda^i V, y \in \Lambda^j W$.

We now define operators α_V, α_W and $\alpha_{V\oplus W}$ on the complexified exterior algebras $\Lambda^*V \otimes \mathbb{C}, \Lambda^*W \otimes \mathbb{C}$ and $\Lambda^*(V \oplus W) \otimes \mathbb{C}$ respectively using the formula from the lectures

$$\alpha_V(x) = (i)^{p(p-1)+\frac{n}{2}} *_V(x)$$

for $x \in \Lambda^p(V) \otimes \mathbb{C}$, and similarly for W and $V \oplus W$.

(b) Prove that

$$\alpha_{V \oplus W}(x \wedge y) = \alpha_V(x) \wedge \alpha_W(y)$$

for all $x \in \Lambda^i V \otimes \mathbb{C}, y \in \Lambda^j W \otimes \mathbb{C}$.

(c) Show that $\alpha^2 = \mathrm{Id}$. So the exterior algebra $\Lambda^* V \otimes \mathbb{C}$ decomposes as $\Lambda^*(V) \otimes \mathbb{C} = \Lambda^+ V \oplus \Lambda^- V$ where $\Lambda^\pm$ denotes the eigenspaces of α with eigenvalues ± 1, and similarly for W and $V \oplus W$. Deduce that there is an isomorphism of $\mathbb{Z}/2$ graded vector spaces

$$\Lambda^*(V \oplus W) \otimes \mathbb{C} \cong (\Lambda^* V \otimes \mathbb{C}) \otimes (\Lambda^* W \otimes \mathbb{C}).$$

Question 15. (a) Now let $E, F \to X$ be real vector bundles equipped with metrics. Let n and m be the ranks of E and F respectively. Using the above define the Hodge star operators

$$*_E : \Lambda^p E \to \Lambda^{n-p} E, *_F : \Lambda^p F \to \Lambda^{m-p} F, *_{E \oplus F} : \Lambda^p(E \oplus F) \to \Lambda^{n+m-p}(E \oplus F)$$

and the corresponding α operators

$$\alpha_E : \Lambda^p E \otimes \mathbb{C} \to \Lambda^{n-p} E \otimes \mathbb{C}, \alpha_F : \Lambda^p F \otimes \mathbb{C} \to \Lambda^{m-p} F \otimes \mathbb{C}, \alpha_{E \oplus F} : \Lambda^p(E \oplus F) \otimes \mathbb{C} \to \Lambda^{n+m-p}(E \oplus F) \otimes \mathbb{C}.$$

(b) Prove that

$$\Lambda^*(E \oplus F) \otimes \mathbb{C} \cong (\Lambda^* E \otimes \mathbb{C}) \otimes (\Lambda^* F \otimes \mathbb{C})$$

as $\mathbb{Z}/2$-graded bundles, with the grading induced by the decomposition of the bundles into the eigensubbundles of the α operators with eigenvalues ± 1.

(c) We now let $\Lambda^\pm$ denote the eigensubbundles of $\Lambda^*(-) \otimes \mathbb{C}$ which correspond to the eigenvalues ± 1 of the α operator. Using the splitting principle prove that

$$\mathrm{ch}(\Lambda^+ E - \Lambda^- E)$$

can be obtained as follows: express the product

$$\prod_{i=1}^{n/2} (e^{-x_i} - e^{x_i})$$

in terms of the elementary symmetric polynomials in x_i and then substitute $c_i(E \otimes \mathbb{C})$ for the i-th elementary symmetric polynomial.

Question 16. (a) Mimicking the proof for the de Rham operator from the previous section prove that the signature of a smooth compact orientable manifold M of even dimension n (and without boundary) can be obtained as follows: express the product

$$\prod_{i=1}^{n/2} \frac{x_i}{\tanh(x_i/2)}$$

in terms of the elementary symmetric polynomials in x_i^2, then substitute $p_i(E)$ for the i-th elementary symmetric polynomial and evaluate the result on the fundamental class of M.

(b) Show that in part (a) instead of $\prod_{i=1}^{n/2} \frac{x_i}{\tanh(x_i/2)}$ one could use $\prod_{i=1}^{n/2} \frac{x_i}{\tanh x_i}$.

(c) Find an explicit formula for the signature of M in terms of the Pontrjagin classes if $n = 4$ and 8.