An introduction to elliptic operators, HSE/MiM, 2016-2017 Problem sheet 6

Clifford algebras

Recall that if V is a vector space over a field k and $q : V \to k$ is a quadratic form, then the *Clifford algebra* $\mathrm{Cl}(V, q)$ is the quotient of the tensor algebra

$$\bigoplus_{n \in \mathbb{Z}_{\geq 0}} V^{\otimes n}$$

by the two sided ideal generated by $x \otimes x + q(x)1$. Here $1 \in V^{\otimes 0} = k$. In this problem sheet we only consider fields of characteristic $\neq 2$. So we can construct the *polarisation* B of q: this is a symmetric bilinear form on V given by $2B(x, y) = q(x + y) - q(x) - q(y)$. A basis $e_1, \dots, e_n$ of V is *orthogonal* iff $B(e_i, e_j) = 0$ once $i \neq j$.

We will use $\cdot$ to denote the multiplication in $\mathrm{Cl}(V, q)$.

Question 1. (a) Show that if V, q are as above and $e_1, \dots, e_n$ is an orthogonal basis of V, then $\mathrm{Cl}(V, q)$ may also be defined as the associative algebra with a unit with generators $e_1, \dots, e_n$ and relations $e_i \cdot e_j = -e_j \cdot e_i$ for $i \neq j$ and $e_i^2 = -q(e_i)$.

(b) Show that $e_{i_1} \cdot \dots \cdot e_{i_m}$ for all sequences $1 \leq i_1 < \dots < i_m \leq n$ form a k-basis of $\mathrm{Cl}(V, q)$. Note that the empty sequence is allowed and it corresponds to the unit 1 of the Clifford algebra.

To get the hang of what Clifford algebras are we consider a few examples. We set $k = \mathbb{R}, V = \mathbb{R}^n$ with the standard basis $e_1, \dots, e_n$ and let C_n^+, respectively C_n^- to be the Clifford algebra constructed using the form $\sum_{i=1}^n x_i^2$, respectively $-\sum_{i=1}^n x_i^2$.

In the rest of this problem sheet $\otimes$ denotes the tensor product over $\mathbb{R}$. Let A, B be two $\mathbb{R}$-algebras. We will write $A \times B$ for the direct product (or direct sum) algebra; its operations are defined component-wise. We will write $\mathrm{M}(n, A)$ to denote the algebra of all $n \times n$ matrices with entries in A.

Question 2. Show that $C_0^\pm \cong \mathbb{R}, C_1^+ \cong \mathbb{C}, C_1^- \cong \mathbb{R} \times \mathbb{R}$ and $C_2^+ \cong \mathbb{H}, C_2^- \cong \mathrm{M}(2, \mathbb{R})$. [These isomorphisms are not required to ba canonical in any meaningful way.]

Note that these examples show that Clifford algebras tend to have a lot of invertible elements.

Question 3. Prove that $C_n^\pm \cong C_2^\pm \otimes C_{n-2}^\mp$. [Again, the isomorphism does not have to be functorial.]

Question 4. (a) Show that $\mathbb{H} \otimes \mathbb{C} \cong \mathrm{M}(2, \mathbb{C})$ and $\mathbb{H} \otimes \mathbb{H} \cong \mathrm{M}(4, \mathbb{R})$.

(b) Show that if A is an $\mathbb{R}$-algebra, then

$$(\mathbb{R} \times \mathbb{R}) \otimes A \cong A \times A, \mathrm{M}(n, R) \otimes A \cong \mathrm{M}(n, A), \mathrm{M}(n, R) \otimes \mathrm{M}(m, \mathbb{R}) \cong \mathrm{M}(mn, \mathbb{R}).$$

Question 5. (a) Using the results above prove that up to isomorphism $C_n^\pm$ are given by the following table for $n \leq 8$:

n	0	1	2	3	4	5	6	7	8
C_n^+	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \times \mathbb{H}$	$\mathrm{M}(2, \mathbb{H})$	$\mathrm{M}(4, \mathbb{C})$	$\mathrm{M}(8, \mathbb{R})$	$\mathrm{M}(8, \mathbb{R}) \times \mathrm{M}(8, \mathbb{R})$	$\mathrm{M}(16, \mathbb{R})$
C_n^-	$\mathbb{R}$	$\mathbb{R} \times \mathbb{R}$	$\mathrm{M}(2, \mathbb{R})$	$\mathrm{M}(2, \mathbb{C})$	$\mathrm{M}(2, \mathbb{H})$	$\mathrm{M}(2, \mathbb{H}) \times \mathrm{M}(2, \mathbb{H})$	$\mathrm{M}(4, \mathbb{H})$	$\mathrm{M}(8, \mathbb{C})$	$\mathrm{M}(16, \mathbb{R})$

(b) Show that for $n \geq 8$ we have $C_n^\pm \cong \mathrm{M}(16, C_{n-8}^\pm)$.