An introduction to elliptic operators, HSE/MiM, 2016-2017 Problem sheet 7

Representations of Clifford algebras and vector fields on spheres

Question 1. Let A be a finite-dimensional algebra over a field k. A *minimal* representation of A is a representation of A over k of minimal k-dimension. Note that A may have several non-isomorphic minimal representations.

(a) Show that if A is a division algebra then the action of A on itself is a minimal representation.

(b) Show that if B is a finite-dimensional division algebra over k and $A = \mathrm{M}(n, B)$, the algebra of all $n \times n$ matrices over B, then the action of A on B^n is a minimal representation.

(c) Show that B is a finite-dimensional division algebra over k, then a minimal representation of $A = \mathrm{M}(n, B) \times \mathrm{M}(n, B)$ is obtained by projecting on one of the factors and then letting $\mathrm{M}(n, B)$ act on B^n.

Question 2. Let $d_n^\pm$ be the dimension of a minimal representation of $C_n^\pm$.

(a) Using question 1 and also what you showed in the previous problem sheet prove that $d_n^\pm$ are given by the following table for $n = 0, \ldots, 8$:

n	0	1	2	3	4	5	6	7	8
d_n^+	1	2	4	4	8	8	8	8	16
d_n^-	1	1	2	4	8	8	16	16	16

(b) Show also that $d_{n+8}^\pm = 2^4 d_n^\pm$.

From now on we will only use d_n^+, so we set $d_n = d_n^+$.

Question 3. (a) Suppose $C_n^+ = \mathrm{Cl}(\mathbb{R}^n, \sum x_i^2)$ acts on the space $\mathbb{R}^m$. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$.

(a) Show that $\mathbb{R}^m$ has a scalar product $(-,-)$ such that Clifford multiplication by each e_i is orthogonal.

(b) Show that then the unit sphere $S^{m-1} \subset \mathbb{R}^m$ has n everywhere pairwise orthogonal tangent vector fields.

(c) Deduce the Hurwitz-Radon theorem: for $m \geq 2$ the sphere S^{m-1} has at least

$$v_m = \max\{n \mid d_n \text{ divides } m\}$$

linearly independent vector fields. Calculate v_m for $m = 0, \ldots, 8$.

Remark. In fact the Radon-Hurwitz lower bound on the number of everywhere linearly independent vector fields on spheres is exact: S^{m-1} cannot have any more such vector fields than what is given by the construction above. This was proved by J. F. Adams. There are several versions of the proof and all involve quite a lot of algebraic topology. I'm not sure we'll get there in this course.

Involutions, Pin and Spin

Let V be a finite-dimensional vector space over a field k and let q be a quadratic form on V. The Clifford algebra $\mathrm{Cl}(V, q)$ has a natural involution, which we denote α. It is induced by the automorphism $x \mapsto -x$ of the form q. The eigenspaces of α with eigenvalues 1 and -1 are denoted $\mathrm{Cl}^0(V, q)$, respectively $\mathrm{Cl}^1(V, q)$. Elements of $\mathrm{Cl}^0(V, q)$, respectively $\mathrm{Cl}^1(V, q)$, will be called *even*, respectively *odd*.

We also have an inti-involution of $\mathrm{Cl}(V, q)$, called the *transpose* and denoted $x \mapsto x^t$. Recall that an *anti-endomorphism* of an algebra A over k is a k-linear self-map $f : A \to A$ which reverses the order of factors in all products, meaning that $f(ab) = f(b)f(a)$ for all $a, b \in A$. An anti-involution is an anti-endomorphism which is squares to the identity. The transpose of $\mathrm{Cl}(V, q)$ is induced by the anti-involution

$$v_1 \otimes \cdots \otimes v_k \mapsto v_k \otimes \cdots \otimes v_1$$

of the tensor algebra. The eigenspaces of the transpose with eigenvalues 1 and -1 are denoted $\mathrm{Cl}^+(V, q)$, respectively $\mathrm{Cl}^-(V, q)$.

Question 4. (a) Show that the transpose is indeed well defined.

(b) Show that α makes $\mathrm{Cl}(V, q)$ into a $\mathbb{Z}/2$-graded algebra, i.e., that

$$\mathrm{Cl}^a(V, q) \cdot \mathrm{Cl}^b(V, q) \subset \mathrm{Cl}^{a+b}(V, q)$$

where $a, b \in \mathbb{Z}/2$.

(c) Describe α and the transpose explicitly when $k = \mathbb{R}$ and the Clifford algebra is C_n^+ for $n = 0, 1, 2, 3$.

We denote the group of invertible elements of $\mathrm{Cl}(V, q)$ as $\mathrm{Cl}^*(V, q)$. We set $\mathrm{Pin}(V, q)$ to be the subgroup of $\mathrm{Cl}^*(V, q)$ generated by all $x \in V$ such that $q(x) \in \{\pm 1\}$. We also set $\mathrm{Spin}(V, q) = \mathrm{Pin}(V, q) \cap \mathrm{Cl}^0(V, q)$.

Question 5. (a) Show that, more explicitly, $\mathrm{Pin}(V, q)$ is the set of all products $x_1 \cdot \cdots \cdot x_l$ with all $x_i \in V$ such that $q(x_i) \in \{\pm 1\}$. The case $l = 0$ is also allowed; the resulting product does not contain any $x \in V$ and is equal to 1 or -1. Moreover, show that such a product $\in \mathrm{Cl}^l(V, q)$ and so $\mathrm{Spin}(V, q)$ is the set of all products as above with an even number of factors.

(b) Now let $x \in V$ be a vector such that $q(x) \in \{\pm 1\}$. Show that the map $y \mapsto x \cdot y \cdot x^{-1}$ restricted to $V \subset \mathrm{Cl}(V, q)$ is minus the reflection in the orthogonal complement of the vector subspace of V spanned by x.

Given a $g \in \mathrm{Pin}(V, q)$ we define a map $R_g : V \to V$ by the formula

$$x \mapsto \alpha(g) x g^{-1}.$$

This is clearly a representation of $\mathrm{Pin}(V, q)$ and we want to calculate its kernel and image. We start with the image.

Question 6. (a) Prove the Cartan-Dieudonné theorem: every element of $\mathrm{O}(V, q)$ is a product of reflections and every element of $\mathrm{SO}(V, q)$ is a product of an even number of reflections. You may assume that $k = \mathbb{R}$ and the quadratic form q is positive definite, although the theorem is true without these assumptions.

(b) Deduce that the map $g \mapsto R_g$ takes $\mathrm{Pin}(V, q)$, respectively $\mathrm{Spin}(V, q)$, surjectively to $\mathrm{O}(V, q)$, respectively to $\mathrm{SO}(V, q)$.

We now calculate the kernel.

Question 7. Show that an even element g of $\mathrm{Pin}(V, q)$ cannot commute with every $x \in V$ unless $g \in \{\pm 1\}$.

(b) Show that an odd element g of $\mathrm{Pin}(V, q)$ cannot anto-commute with every $x \in V$.

(c) Deduce that the kernel of the representation $g \mapsto R_g$ of $\mathrm{Pin}(V, q)$ is $\{pm1\}$.

Remark. Spin groups are so called because they have something to do with spins of particles. I'm not sure what these are (either spins or particles), so I'll say no more about it, but Pin groups have nothing to do with pins. Instead, the rationale behind the name is this: SO is to O as Spin is to what? Pin, of course! I think it was Paul Dirac who came up with both the terminology and the explanation.