An introduction to elliptic operators, HSE/MiM, 2016-2017 Problem sheet 8

Clifford algebras continued

Question 1 Let V be a finite-dimensional vector space over a field k of characteristic $\neq 2$ and let $q : V \to k$ be a quadratic form. Prove that the Clifford algebra $\mathrm{Cl}(V, q)$ is isomorphic as a vector space to the exterior algebra $\Lambda^* V$ functorially in V (in particular, the isomorphism depends only on V and not on q).

Question 2. Let k, V, q be as above. You may assume that $k = \mathbb{R}$ and q is positive definite. Set $n = \dim_k V$. Using the isomorphism from question 1, describe the Hodge star operator

$$* : \Lambda^i(V) \to \Lambda^{n-i} V$$

in terms of the Clifford multiplication. [Hint: choose an orthogonal basis and consider the product of all its elements; there might be sign issues to take care of.]

Question 3. Let k be as above and let (V, q_1) and (W, q_2) be finite-dimensional k-vector spaces equipped with quadratic forms q_1, q_2. Let $q_1 \oplus q_2$ be the direct sum of q_1 and q_2. Set $n = \dim_k V, m = \dim_k W$ and let $*_V, *_W, *_{V\oplus W}$ be the Hodge star operators on V, W and $V \oplus W$ equipped with the quadratic forms q_1, q_2 and $q_1 \oplus q_2$ respectively. We assume that both m and n are even.

(a) Prove that

$$*_{V\oplus W}(x \wedge y) = (-1)^{ij} *_V (x) \wedge *_W(y)$$

for all $x \in \Lambda^i V, y \in \Lambda^j W$.

We now define operators α_V, α_W and $\alpha_{V\oplus W}$ on the complexified exterior algebras $\Lambda^* V \otimes \mathbb{C}, \Lambda^* W \otimes \mathbb{C}$ and $\Lambda^*(V \oplus W) \otimes \mathbb{C}$ respectively using the formula from the lectures

$$\alpha_V(x) = (i)^{p(p-1)+\frac{n}{2}} *_V (x)$$

for $x \in \Lambda^p(V) \otimes \mathbb{C}$, and similarly for W and $V \oplus W$.

(b) Prove that

$$\alpha_{V\oplus W}(x \wedge y) = \alpha_V(x) \wedge \alpha_W(y)$$

for all $x \in \Lambda^i V \otimes \mathbb{C}, y \in \Lambda^j W \otimes \mathbb{C}$.

(c) Show that $\alpha^2 = \mathrm{Id}$. So the exterior algebra $\Lambda^* V \otimes \mathbb{C}$ decomposes as $\Lambda^*(V) \otimes \mathbb{C} = \Lambda^+ V \oplus \Lambda^- V$ where $\Lambda^\pm$ denotes the eigenspaces of α with eigenvalues ± 1, and similarly for W and $V \oplus W$. Deduce that there is an isomorphism of $\mathbb{Z}/2$ graded vector spaces

$$\Lambda^*(V \oplus W) \otimes \mathbb{C} \cong (\Lambda^* V \otimes \mathbb{C}) \otimes (\Lambda^* W \otimes \mathbb{C}).$$

The L-genus

Question 4. (a) Now let $E, F \to X$ be real vector bundles equipped with metrics. Let n and m be the ranks of E and F respectively. Using the above define the Hodge star operators

$$*_E : \Lambda^p E \to \Lambda^{n-p} E, *_F : \Lambda^p F \to \Lambda^{m-p} F, *_{E\oplus F} : \Lambda^p(E \oplus F) \to \Lambda^{n+m-p}(E \oplus F)$$

and the corresponding α operators

$$\alpha_E : \Lambda^p E \otimes \mathbb{C} \to \Lambda^{n-p} E \otimes \mathbb{C}, \alpha_F : \Lambda^p F \otimes \mathbb{C} \to \Lambda^{m-p} F \otimes \mathbb{C}, \alpha_{E\oplus F} : \Lambda^p(E \oplus F) \otimes \mathbb{C} \to \Lambda^{n+m-p}(E \oplus F) \otimes \mathbb{C}.$$

(b) Prove that

$$\Lambda^*(E \oplus F) \otimes \mathbb{C} \cong (\Lambda^* E \otimes \mathbb{C}) \otimes (\Lambda^* F \otimes \mathbb{C})$$

as $\mathbb{Z}/2$-graded bundles, with the grading induced by the decomposition of the bundles into the eigensubbundles of the α operators with eigenvalues ± 1.

(c) We now let $\Lambda^\pm$ denote the eigensubbundles of $\Lambda^*(-) \otimes \mathbb{C}$ which correspond to the eigenvalues ± 1 of the α operator. Using the splitting principle prove that

$$\mathrm{ch}(\Lambda^+ E - \Lambda^- E)$$

can be obtained as follows: express the product

$$\prod_{i=1}^{n/2}(e^{-x_i} - e^{x_i})$$

in terms of the elementary symmetric polynomials in x_i and then substitute $c_i(E\otimes\mathbb{C})$ for the i-th elementary symmetric polynomial.

Question 5. (a) Mimicking the proof for the de Rham operator from problem sheet 4 on characteristic classes prove that the signature of a smooth compact orientable manifold M of even dimension n (and without boundary) can be obtained as follows: express the product

$$\prod_{i=1}^{n/2}\frac{x_i}{\tanh(x_i/2)}$$

in terms of the elementary symmetric polynomials in x_i^2, then substitute $p_i(E)$ for the i-th elementary symmetric polynomial and evaluate the result on the fundamental class of M.

(b) Show that in part (a) instead of $\prod_{i=1}^{n/2}\frac{x_i}{\tanh(x_i/2)}$ one could use $\prod_{i=1}^{n/2}\frac{x_i}{\tanh x_i}$.

(c) Find an explicit formula for the signature of M in terms of the Pontrjagin classes if $n=4$ and 8.